INTEGRAL KHUSUS

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Abstracts. In given the lecture for Mathematics or another names is Calculus we must following the based of Recana Perkuliahan Semester (RPS) and therefore we didn't to introduction for student Specific Integrals and therefore we introduction specific integrals.

Abstrak. Dalam melaksanakan tugas perkuliahan, para rekan dosen mengikuti silabus yang tertera pada Recana Perkuliahan Semester (RPS) untuk integral khusus tidak diajarkan. Aplikasi integral inimeliputi:Notasi berindeks, Tensor danChronecker. Dalam rangka menambah wawasan pengetahuan maka penulis merasa perlu membicarakan hal ini. Disini akan dibahas tentang integral-integral khusus dengan menggunakan teknik tensor.

Kata kunci
Integral, Lintasan, Notasi berindeks, Tensor, Chronecker

PENDAHULUAN

Pada tulisan ini akan ditampilkan integral-integral yang jarang dibahas didalam perkuliahan-perkuliahan kalkulus pada umumnya terutama aplikasi-aplikasi indeksnya. Disini penulis dalam penyusunan tulisan ini dilengkapi dengan tuntunan dari Prof. Pantur Silaban, Ph.D. Disamping itu penulis juga sedang berusaha berkonsultasi dengan Prof. Erwin Sucipto, Ph.D di Bethel College, Indiana Amerika Serikat karena beliau dikenal oleh penulis sebagai kakak kelas penulis di ITB, 1973. Dengan misi demi pengembangan penguasaan materi maka penulis memberanikan diri untuk membuat tulisan ini.

METODE

Kajian ini berdasarkan dengan mencoba membuktikan sendiri dari daftar integral dari Literatur
yang berjudul: "Abromowitz, Milton, and Irene A. Stegun, editors, *Handbook of Mathematical Functions With Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, AppliedMathematical Series, 55, U.S. Government Printing Office, Washington, D.C., 1964. Dimana pada buku tersebut tidak terdapat pembuktian secara tertulis tentang tabel integralnyadan penulis memilih berdasarkan pengalaman dari teman-teman dosen yang belumdapat menyelesaikannyasecaraanalitis. Ada 2 jenis pembahasan yaitu dengan menjawab pertanyaan integraldan dengan membuktikan suatu integral pada tabel.

Integral Khusus

Invers dan kebalikkan.

Invers dan kebalikkan adalah dua hal yang berbeda jika dipandang secara matematis. Invers adalah kebalikkan secara fungsi atau kebalikan secara operator. Sedangkan kebalikkan diartikan sebagai fungsi kebalikkan.
Contoh fungsi kebalikkan:
$y = x \xleftrightarrow{fungsi\ kebalikkan} y = \frac{1}{x}$, $y = x^2 \xleftrightarrow{fungsi\ kebalikkan} y = \frac{1}{x^2}$,

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$y = \sin x \xleftrightarrow{fungsi\ kebalikkan} y = \csc x$, $y = \cos x \xleftrightarrow{fungsi\ kebalikkan} y = \sec x$
$y = \tan x \xleftrightarrow{fungsi\ kebalikkan} y = \cot x$, $y = C^x \xleftrightarrow{fungsi\ kebalikkan} y = C^{-x}$
$y = 10^x \xleftrightarrow{fungsi\ kebalikkan} y = 10^{-x}$, $y = e^x \xleftrightarrow{fungsi\ kebalikkan} y = e^{-x}$ dst
Contoh fungsi invers:
$y = x \xleftrightarrow{invers} x = y$, $y = \frac{1}{x} \xleftrightarrow{invers} x = \frac{1}{y}$, $y = x + C \xleftrightarrow{invers} x = y - C$
$y = x^2 \xleftrightarrow{invers} x = \pm\sqrt{y}$, $y = \sin x \xleftrightarrow{invers} x = \sin^{-1} y$, $y = \cos x \xleftrightarrow{invers} x = \cos^{-1} y$
$y = \tan x \xleftrightarrow{invers} x = \tan^{-1} y$, $y = \operatorname{cosec} x \xleftrightarrow{invers} x = \operatorname{cosec}^{-1} y$, $y = \sec x \xleftrightarrow{invers} x = \sec^{-1} y$
$y = \operatorname{ctan} x \xleftrightarrow{invers} x = \operatorname{ctan}^{-1} y$, $y = C^x \xleftrightarrow{invers} x = \log_C y$, $y = 10^x \xleftrightarrow{invers} x = \log|y|$
$y = e^x \xleftrightarrow{invers} x = \ln|y|$, dst.
Karena bilangan 10 dan *e* keduanya lebih besar dari nol, maka *y*> 0 sehingga untuk menghindari harga negatip, maka harga *y* diberi tanda mutlak seperti $|y|$. Untuk bilangan dasar *C* tidak perlu diberi tanda harga mutlak karena bisa saja *C* berharga negatip.
Contoh invers operator:
$\frac{d}{dx}[C + F(x)] = f(x) \xleftrightarrow{invers} \int f(x)\,dx = C + F(x)$, $mengembang \xleftrightarrow{invers} menciut$
$stationer \xleftrightarrow{invers} gerak\ periodik$ dst.
Pada contoh terakhir berarti bila diketahui turunan suatu fungsi maka akan diketahui juga inversnya berupa fungsi asal (sebelum diturunkan) yang diistilahkan sebagai ***"integrasi"*** fungsi.

Sifat-sifat turunan fungsi
Pembahasan: $\frac{0}{0}$ = tak tentu.
Agar menghasilkan hal tertentu maka jangan nol akan tetapi mendekati nol atau populernya limit nol = $\lim_{\Delta y \to 0} \Delta y$, jika: $y = f(x)$, maka: $\frac{\lim_{\Delta y \to 0} \Delta y}{\lim_{\Delta x \to 0} \Delta x} = \frac{\lim_{\Delta x \to 0} f(x+\Delta x) - f(x)}{\lim_{\Delta x \to 0} \Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$. Selanjutnya $\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{d}{dx}\{f(x)\} = \frac{d}{dx}y = \frac{dy}{dx}$ = tertentu. Sehingga didefinisikanlah: $\frac{d}{dx}\{f(x)\} = lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{dy}{dx} = y' = y^{(1)}$, ($\frac{d}{dx}\{f(x)\}$ = turunan $f(x)$ terhadap x dan bentuk invers nya disebut "integral $f(x)$ terhadap x" dengan lambang: $\int f(x)dx$.
Sehingga dapat juga diturunkan fungsi-fungsi: $f(x)g(x)$, $\frac{f(x)}{g(x)}$ dan $(f \circ g)(x)$

- $\frac{d}{dx}\{f(x)g(x)\} = f'(x)g(x) + f(x)g'(x)$atau: $\{fg\}' = f'g + fg'$
- $\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{\frac{df(x)}{dx}(x)g(x) - f(x)\frac{dg(x)}{dx}}{\{g(x)\}^2}$, atau:$\left\{\frac{f}{g}\right\}' = \frac{f'g - fg'}{g^2}$
- $\frac{d}{dx}\{(f \circ g)(x)\} = \frac{d}{dx}[f\{g(x)\}] = \frac{dg(x)}{dx}\frac{df\{g(x)\}}{dg(x)}$***"aturan rantai"***.

Turunan dari fungsi $y = \frac{1+\sin x}{\cos x}$ dan $y = \frac{1+\cos x}{\sin x}$. Dengan memakai sifat diatas maka:
$\frac{dy}{dx} = \frac{y}{\cos x}$,jadi:$\int \frac{dx}{\cos x} = ln\left|C\left(\frac{1+\sin x}{\cos x}\right)\right|$
$\frac{dy}{dx} = -\frac{y}{\sin x}$, jadi:$\int \frac{dx}{\sin x} = ln\left|\frac{C\sin x}{1+\cos x}\right|$

HASIL DAN PEMBAHASAN

Teknik khusus pengintegralan

Untuk mencari jawaban suatupengintegralan dapat ditempuh dengan beberapa cara yang dinamakan teknik pengintegralan. Ada beberapa teknik pengintegralan yaitucara parsial, substitusi dan cara penderetan. Ketiga cara ini dipakai untuk teknik-teknik khusus pengintegralan.

Contoh:

1. Tentukanlah integral berikut: (a). $\int \frac{dx}{\cos^3 x}$ dan (b). $\int \frac{dx}{\sin^3 x}$

   Jawab:

   (a) $\int \frac{dx}{\cos^3 x} = \int \frac{1}{\cos^2 x}\frac{1}{\cos x} dx = \tan x\,(\cos x)^{-1} - \int \tan x \frac{d}{dx}(\cos x)^{-1}\, dx$
   $= \frac{\sin x}{\cos^2 x} - \int \frac{\sin x}{\cos x}(-1)(\cos x)^{-2}(-\sin x)\, dx = \frac{\sin x}{\cos^2 x} - \int \frac{\sin^2 x}{\cos^3 x} dx$
   $= \frac{\sin x}{\cos^2 x} - \int \frac{1-\cos^2 x}{\cos^3 x} dx = \frac{\sin x}{\cos^2 x} - \int \frac{dx}{\cos^3 x} + \int \frac{\cos^2 x}{\cos^3 x} dx$
   $= \frac{\sin x}{\cos^2 x} - \int \frac{dx}{\cos^3 x} + \int \frac{dx}{\cos x} = \frac{\sin x}{\cos^2 x} - \int \frac{dx}{\cos^3 x} + \ln\left|\frac{1+\sin x}{\cos x}\right| + \ln C$
   $= \frac{\sin x}{\cos^2 x} - \int \frac{dx}{\cos^3 x} + \ln C\left|\frac{1+\sin x}{\cos x}\right|$, atau: $\int \frac{dx}{\cos^3 x} = \frac{\sin x}{\cos^2 x} - \int \frac{dx}{\cos^3 x} + \ln C\left|\frac{1+\sin x}{\cos x}\right|$

   Atau: $2\int \frac{dx}{\cos^3 x} = \frac{\sin x}{\cos^2 x} + \ln C\left|\frac{1+\sin x}{\cos x}\right|$.

   Sehingga: $\int \frac{dx}{\cos^3 x} = \frac{\sin x}{2\cos^2 x} + \frac{1}{2}\ln\left\{\ln C\left|\frac{1+\sin x}{\cos x}\right|\right\} = \frac{\sin x}{2\cos^2 x} + \ln\sqrt{C\left|\frac{1+\sin x}{\cos x}\right|}$

   Jadi: $\int \frac{dx}{\cos^3 x} = \frac{\sin x}{2\cos^2 x} + \ln\sqrt{C\left|\frac{1+\sin x}{\cos x}\right|}$

   Dengan cara yang sama dapat ditunjukkan bahwa:

   (b) $\int \frac{dx}{\sin^3 x} = -\frac{\cos x}{2\sin^2 x} + \ln\sqrt{C\left|\frac{\sin x}{1+\cos x}\right|}$

2. Tentukanlah integral berikut: (a). $\int \frac{dx}{\cos^5 x}$ dan (b). $\int \frac{dx}{\sin^5 x}$

   Jawab:

   (a) $\int \frac{dx}{\cos^5 x} = \int \frac{1}{\cos^2 x}\frac{1}{\cos^3 x} dx = \tan x\,(\cos x)^{-3} - \int \tan x \frac{d}{dx}\{(\cos x)^{-3}\}\, dx$
   $= \frac{\sin x}{\cos^4 x} - \int \frac{\sin x}{\cos x}(-3)(\cos x)^{-4}(-\sin x)\, dx = \frac{\sin x}{\cos^4 x} - 3\int \frac{\sin^2 x}{\cos^5 x} dx$
   $= \frac{\sin x}{\cos^4 x} - 3\int \frac{1-\cos^2 x}{\cos^5 x} dx = \frac{\sin x}{\cos^4 x} - 3\int \frac{1}{\cos^5 x} dx + 3\int \frac{dx}{\cos^3 x}$
   $= \frac{\sin x}{\cos^4 x} - 3\int \frac{1}{\cos^5 x} dx + 3\left(\frac{\sin x}{2\cos^2 x} + \ln\sqrt{C\left|\frac{1+\sin x}{\cos x}\right|}\right)$atau4 $\int \frac{dx}{\cos^5 x}$
   $= \frac{\sin x}{\cos^4 x} + \frac{3\sin x}{2\cos^2 x} + 3\ln\sqrt{C\left|\frac{1+\sin x}{\cos x}\right|}\int \frac{dx}{\cos^5 x} = \frac{\sin x}{4\cos^4 x} + \frac{3\sin x}{8\cos^2 x} + \frac{3}{4}\ln\sqrt{C\left|\frac{1+\sin x}{\cos x}\right|}$

   Jadi: $\int \frac{dx}{\cos^5 x} = \frac{\sin x}{4\cos^4 x} + \frac{3\sin x}{8\cos^2 x} + \frac{3}{4}\ln\sqrt{C\left|\frac{1+\sin x}{\cos x}\right|}$

   Dengan cara yang sama dapat dibuktikan bahwa:

   (b) $\int \frac{dx}{\sin^5 x} = -\frac{\cos x}{4\sin^4 x} - \frac{3}{8}\frac{\cos x}{\sin^2 x} + \frac{3}{4}\ln\sqrt{C\left|\frac{\sin x}{1+\cos x}\right|}$

3. Tentukanlah: (a). $\int \frac{a}{x+b} dx$, (b). $\int \frac{a}{x^2-b^2} dx$, (c). $\int \frac{a}{x^2+b^2} dx$ dan (d). $\int \frac{a}{x^2+bx+c} dx$

   Jawab:

(a) $\int \frac{a}{x+b} dx = a\int \frac{dx}{x+b} = a\int \frac{dw}{w} = a(C + \ln|w|) = D + a\ln|x+b| = D + \ln|x+b|^a$
Jadi: $\int \frac{a}{x+b} dx = D + \ln|x+b|^a$

(b) $\int \frac{a}{x^2-b^2} dx = a\int \frac{dx}{x^2-b^2} = a\int \frac{dx}{(x+b)(x-b)} = a\int \left\{\frac{A}{x+b} + \frac{B}{x-b}\right\} dx$
$= a\int \frac{A(x-b)+B(x+b)}{(x+b)(x-b)} dx = a\int \frac{Ax-bA+Bx+bB}{(x+b)(x-b)} dx = a\int \frac{(A+B)x-b(A+B)}{(x+b)(x-b)} dx,$
maka: $B + A = 0$ dan $b(B - A) = a$
$A = -B$ dan$B - A = \frac{a}{b} = B - (-B) = 2B = \frac{a}{b}$, maka: $B = \frac{a}{2b}$ dan $A = -\frac{a}{2b}$
Sehingga: $a\int \left\{\frac{A}{x+b} + \frac{B}{x-b}\right\} dx = a\int \left\{-\frac{\frac{a}{2b}}{x+b} + \frac{\frac{a}{2b}}{x-b}\right\} dx$
$= -\frac{a^2}{2b}\int \frac{dx}{(x+b)} + \frac{a^2}{2b}\int \frac{dx}{(x-b)} dx = -\frac{a^2}{2b}\ln|x+b| + \frac{a^2}{2b}\ln|x-b| + C$
$= \frac{a^2}{2b}\ln\left|\frac{x-b}{x+b}\right| + C = \ln\left|\frac{x-b}{x+b}\right|^{\frac{a^2}{2b}} + C$. Jadi: $\int \frac{a}{x^2-b^2} dx = \ln\left|\frac{x-b}{x+b}\right|^{\frac{a^2}{2b}} + C$

(c) $\int \frac{a}{x^2+b^2} dx = \frac{a}{b^2}\int \frac{b^2}{x^2+b^2} dx$, dimana: $\frac{b}{\sqrt{x^2+b^2}} = \sin\theta$, $x = b\cot\theta$, $dx = -\frac{b}{\sin^2\theta} d\theta$, maka: $\frac{a}{b^2}\int \frac{b^2}{x^2+b^2} dx = \frac{a}{b^2}\int \sin^2\theta\left(-\frac{b}{\sin^2\theta} d\theta\right) dx = -\frac{a}{b}\int d\theta$
$= C - \frac{a}{b}\cot^{-1}\left(\frac{x}{b}\right)$
Jadi: $\int \frac{a}{x^2+b^2} dx = C - \frac{a}{b}\cot^{-1}\left(\frac{x}{b}\right)$

(d) $\int \frac{a}{x^2+bx+c} dx = a\int \frac{dx}{x^2+2\left(\frac{b}{2}\right)x+\frac{b^2}{4}-\frac{b^2}{4}+c} = a\int \frac{dx}{\left\{x^2+2\left(\frac{b}{2}\right)x+\frac{b^2}{4}\right\}-\left(\frac{b^2-4c}{4}\right)}$
$= a\int \frac{dx}{\left(x+\frac{b}{2}\right)^2-\left(\frac{\sqrt{b^2-4c}}{2}\right)^2} = a\int \frac{du}{u^2-\beta^2} = a\int \frac{du}{(u+\beta)(u-\beta)} = a\int \left(\frac{A}{u+\beta} + \frac{B}{u-\beta}\right) du$
$= a\int \frac{A(u-\beta)+B(u+\beta)}{(u+\beta)(u-\beta)} du = a\int \frac{Au-A\beta+Bu+B\beta}{u^2-\beta^2} du = a\int \frac{(A+B)u-(A-B)\beta}{u^2-\beta^2} du,$
maka:
$A + B = 0$ atau: $A = -B$ dan $(A - B)\beta = -1$, atau: $A - B = -\frac{1}{\beta}$,
sehingga:
$(-B) - B = -2B = -\frac{1}{\beta}$, atau: $B = \frac{1}{2\beta}$ dan $A = -\frac{1}{2\beta}$
Maka: $a\int \left(\frac{A}{u+\beta} + \frac{B}{u-\beta}\right) du = a\int \left(\frac{-\frac{1}{2\beta}}{u+\beta} + \frac{\frac{1}{2\beta}}{u-\beta}\right) du = \frac{a}{2\beta}\int \left(\frac{1}{u-\beta} - \frac{1}{u+\beta}\right) du$
$= \frac{a}{2\beta}\int \frac{du}{u-\beta} - \frac{a}{2\beta}\int \frac{du}{u+\beta} = \frac{a}{2\beta}(\ln|u-\beta| - \ln|u+\beta|) = \frac{a}{2\beta}\ln\left|\frac{u-\beta}{u+\beta}\right| + C$
$= \frac{a}{2\frac{\sqrt{b^2-4c}}{2}}\ln\left|\frac{\frac{2u-\sqrt{b^2-4c}}{2}}{\frac{2u+\sqrt{b^2-4c}}{2}}\right| + C = \frac{a}{\sqrt{b^2-4c}}\ln\left|\frac{2u-\sqrt{b^2-4c}}{2u+\sqrt{b^2-4c}}\right| + C$
$= \ln\left|\frac{2u-\sqrt{b^2-4c}}{2u+\sqrt{b^2-4c}}\right|^{\sqrt{b^2-4c}} + C = \ln\left|\frac{2\left(x+\frac{b}{2}\right)-\sqrt{b^2-4c}}{2\left(x+\frac{b}{2}\right)+\sqrt{b^2-4c}}\right|^{\sqrt{b^2-4c}} + C$
$= \ln\left|\frac{2\left(\frac{2x+b}{2}\right)-\sqrt{b^2-4c}}{2\left(\frac{2x+b}{2}\right)+\sqrt{b^2-4c}}\right|^{\sqrt{b^2-4c}} + C = \ln\left|\frac{2x+b-\sqrt{b^2-4c}}{2x+b+\sqrt{b^2-4c}}\right|^{\sqrt{b^2-4c}} + C$
Jadi: $\int \frac{a}{x^2+bx+c} dx = \ln\left|\frac{2x+b-\sqrt{b^2-4c}}{2x+b+\sqrt{b^2-4c}}\right|^{\sqrt{b^2-4c}} + C$

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4. Tentukanlah integral berikut:
   (a). $\int \sqrt{a+x}\,dx$, (b). $\int \sqrt{a^2-x^2}\,dx$, (c). $\int \sqrt{x^2+4x+3}\,dx$ dan (d). $\int \sqrt{x^2-a^2}\,dx$
   Jawab:
   (a) $\int \sqrt{a+x}\,dx = \int (a+x)^{\frac{1}{2}}\,dx = \frac{2}{3}\int d\left\{(a+x)^{\frac{3}{2}}\right\} = (a+x)^{\frac{3}{2}} + C$
   Jadi: $\int \sqrt{a+x}\,dx = (a+x)^{\frac{3}{2}} + C$
   (b) $\int \sqrt{a^2-x^2}\,dx = a\int \frac{\sqrt{a^2-x^2}}{a}\,dx$, dimana: $\frac{\sqrt{a^2-x^2}}{a} = \sin\theta, \frac{x}{a} = \cos\theta \to x = a\cos\theta \to dx = -a\sin\theta\,d\theta$
   Maka: $a\int \frac{\sqrt{a^2-x^2}}{a}\,dx = a\int \sin\theta\,(-a\sin\theta\,d\theta) = -a^2\int \sin^2\theta\,d\theta$
   $= -a^2\int \left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right)d\theta = -a^2\left\{\frac{1}{2}\int d\theta - \frac{1}{2}\int \cos 2\theta\,\frac{d(2\theta)}{2}\right\}$
   $= -a^2\left\{\frac{1}{2}\cos^{-1}\left(\frac{x}{a}\right) - \frac{1}{4}(\sin 2\theta)\right\} = -a^2\left\{\frac{1}{2}\cos^{-1}\left(\frac{x}{a}\right) - \frac{1}{4}(2\sin\theta\cos\theta)\right\}$
   $= -a^2\left\{\frac{1}{2}\cos^{-1}\left(\frac{x}{a}\right) - \frac{1}{2}(\sin\theta\cos\theta)\right\} + C = -\frac{a^2}{2}\cos^{-1}\left(\frac{x}{a}\right) + \frac{x\sqrt{a^2-x^2}}{2} + C$
   $= \frac{x\sqrt{a^2-x^2}}{2} - \frac{a^2}{2}\cos^{-1}\left(\frac{x}{a}\right) + C.$
   Jadi: $\int \sqrt{a^2-x^2}\,dx = \frac{x\sqrt{a^2-x^2}}{2} - \frac{a^2}{2}\cos^{-1}\left(\frac{x}{a}\right) + C$
   (e) $\int \sqrt{x^2+4x+3}\,dx = \int \sqrt{\{x^2+2(2)x+4\}-4+3}\,dx$
   $= \int \sqrt{(x+2)^2-1}\,dx$, ambil: $x+2=t$, maka: $\int \sqrt{(x+2)^2-1}\,dx$
   $= \int \sqrt{t^2-1}\,dt$, dimana: $t = \csc\theta = \frac{1}{\sin\theta} = (\sin\theta)^{-1}$, maka: dt
   $= (-1)(\sin\theta)^{-2}\cos\theta\,d\theta = -\frac{\cos\theta}{\sin^2\theta}d\theta \sqrt{t^2-1} = \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}.$
   Sehingga: $\int \sqrt{t^2-1}\,dt = -\int \frac{\cos\theta}{\sin\theta}\frac{\cos\theta}{\sin^2\theta}d\theta = -\int \frac{\cos^2\theta}{\sin^3\theta}d\theta = -\int \frac{1-\sin^2\theta}{\sin^3\theta}d\theta$
   $= -\int \frac{d\theta}{\sin^3\theta} + \int \frac{d\theta}{\sin\theta} = -\ln\sqrt{\left|A\left(\frac{\sin\theta}{1+\cos\theta}\right)\right|} + \ln\left|\frac{B\sin\theta}{1+\cos\theta}\right| = \ln\left|\frac{\frac{B\sin\theta}{1+\cos\theta}}{\sqrt{\left|A\left(\frac{\sin\theta}{1+\cos\theta}\right)\right|}}\right|$
   $= \ln\left|\frac{\frac{\sqrt{B}^2\sqrt{\sin\theta}^2}{\sqrt{1+\cos\theta}^2}}{\frac{\sqrt{A}\sqrt{\sin\theta}}{\sqrt{1+\cos\theta}}}\right| = \ln\left|\frac{B}{\sqrt{A}}\frac{\sqrt{\sin\theta}}{\sqrt{1+\cos\theta}}\right| = \ln\left|C\frac{\sqrt{\frac{1}{t}}}{\sqrt{1+\frac{\sqrt{t^2-1}}{t}}}\right| = \ln\left|C\frac{\frac{1}{\sqrt{t}}}{\frac{\sqrt{t+\sqrt{t^2-1}}}{\sqrt{t}}}\right|$
   $= \ln\left|C\frac{1}{\sqrt{t+\sqrt{t^2-1}}}\right| = \ln\left|C\frac{1}{\sqrt{(x+2)+\sqrt{(x+2)^2-1}}}\right| = \ln\left|C\frac{1}{\sqrt{(x+2)+\sqrt{x^2+4x+3}}}\right|$
   Jadi: $\int \sqrt{x^2+4x+3}\,dx = \ln\left|C\frac{1}{\sqrt{(x+2)+\sqrt{x^2+4x+3}}}\right|$
   (f) $\int \sqrt{x^2-a^2}\,dx = a\int \frac{\sqrt{x^2-a^2}}{a}\,dx.$
   Maka: $\frac{\sqrt{x^2-a^2}}{a} = \cot\theta$, $x = a\csc\theta = a\frac{1}{\sin\theta} = a(\sin\theta)^{-1}$
   $dx = a(-1)(\sin\theta)^{-2}\cos\theta\,d\theta = -a\frac{\cos\theta}{\sin^2\theta}d\theta$, atau: $dx = -a\frac{\cos\theta}{\sin^2\theta}d\theta.$

Sehingga: $a\int \frac{\sqrt{x^2-a^2}}{a}dx = a\int \cot\theta\left(-a\frac{\cos\theta}{\sin^2\theta}d\theta\right) =$
$-a^2\int\frac{\cos\theta}{\sin\theta}\left(\frac{\cos\theta}{\sin^2\theta}d\theta\right)$
$= -a^2\int\frac{\cos^2\theta}{\sin^3\theta}d\theta = -a^2\int\frac{1-\sin^2\theta}{\sin^3\theta}d\theta = -a^2\int\frac{1}{\sin^3\theta}d\theta + a^2\int\frac{d\theta}{\sin\theta}$
$= -a^2\int\frac{d\theta}{\sin^3\theta} + a^2\int\frac{d\theta}{\sin\theta} = \frac{a^2\cos\theta}{2\sin^2\theta} - a^2\ln\sqrt{\left|\frac{C\sin\theta}{1+\cos\theta}\right|} + a^2\ln\left|\frac{C\sin\theta}{1+\cos\theta}\right|$
$= a^2\left(\frac{\frac{\sqrt{x^2-a^2}}{x}}{2\frac{a^2}{x^2}} + \ln\sqrt{\frac{A\frac{a}{x}}{1+\frac{\sqrt{x^2-a^2}}{x}}}\right) = a^2\left(\frac{x\sqrt{x^2-a^2}}{2a^2} + \ln\sqrt{\frac{aA}{x+\sqrt{x^2-a^2}}}\right)$
Jadi: $\int\sqrt{x^2-a^2}\,dx = \frac{x\sqrt{x^2-a^2}}{2} - a^2\ln\frac{\sqrt{x+\sqrt{x^2-a^2}}}{B}$

5. Tentukanlah:(a). $\int\frac{dx}{\sqrt{a+x}}$, (b). $\int\frac{dx}{\sqrt{a^2-x^2}}$, (c). $\int\frac{dx}{\sqrt{x^2-a^2}}$ dan (d). $\int\frac{dx}{\sqrt{x^2+bx+c}}$
   Jawab:
   (a) $\int\frac{dx}{\sqrt{a+x}} = \int\frac{dx}{(a+x)^{\frac{1}{2}}} = \int(a+x)^{-\frac{1}{2}}dx = 2\int d(a+x)^{\frac{1}{2}} = 2(a+x)^{\frac{1}{2}} + C$
   Jadi: $\int\frac{dx}{\sqrt{a+x}} = 2(a+x)^{\frac{1}{2}} + C$
   (b) $\int\frac{dx}{\sqrt{a^2-x^2}} = \frac{1}{a}\int\frac{a}{\sqrt{a^2-x^2}}dx$, dimana: $\frac{a}{\sqrt{a^2-x^2}} = \csc\theta$, $x = a\cos\theta \rightarrow$ $dx = -a\sin\theta\,d\theta$, maka: $\frac{1}{a}\int\frac{a}{\sqrt{a^2-x^2}}dx = \frac{1}{a}\int\csc\theta\,(-a\sin\theta\,d\theta) = -\frac{1}{a}\int\frac{\sin\theta}{\sin\theta}d\theta = -\frac{1}{a}\int d\theta = C - \frac{1}{a}\cos^{-1}\left(\frac{x}{a}\right)$. Jadi: $\int\frac{dx}{\sqrt{a^2-x^2}} = C - \frac{1}{a}\cos^{-1}\left(\frac{x}{a}\right)$
   (c) $\int\frac{dx}{\sqrt{x^2-a^2}} = \frac{1}{a}\int\frac{a}{\sqrt{x^2-a^2}}dx$, dimana: $\frac{a}{\sqrt{x^2-a^2}} = \tan\theta$, $x = a\csc\theta = \frac{a}{\sin\theta} = a(\sin\theta)^{-1}$
   $dx = a(-1)(\sin\theta)^{-2}\cos\theta = -\frac{a\cos\theta}{\sin^2\theta}d\theta$
   Maka: $\frac{1}{a}\int\frac{a}{\sqrt{x^2-a^2}}dx = \frac{1}{a}\int\tan\theta\left(-\frac{a\cos\theta}{\sin^2\theta}d\theta\right) = -\int\frac{\sin\theta}{\cos\theta}\frac{\cos\theta}{\sin^2\theta}d\theta$
   $= -\int\frac{d\theta}{\sin\theta} = -\ln\left\{A\left(\frac{\sin\theta}{1+\cos\theta}\right)\right\} = \ln\left\{A\left(\frac{1+\cos\theta}{\sin\theta}\right)\right\} = \ln\left\{A\left(\frac{1+\frac{\sqrt{x^2-a^2}}{x}}{\frac{a}{x}}\right)\right\}$
   $= \ln\left\{A\left(\frac{\frac{x+\sqrt{x^2-a^2}}{x}}{\frac{a}{x}}\right)\right\} = \ln\left\{A\left(\frac{x+\sqrt{x^2-a^2}}{x}\times\frac{x}{a}\right)\right\} = \ln\left\{A\left(\frac{x+\sqrt{x^2-a^2}}{a}\right)\right\}$
   Jadi: $\int\frac{dx}{\sqrt{x^2-a^2}} = \ln\left\{A\left(\frac{x+\sqrt{x^2-a^2}}{a}\right)\right\}$
   (d) $\int\frac{dx}{\sqrt{x^2+bx+c}} = \int\frac{dx}{\sqrt{x^2+2x\left(\frac{b}{2}\right)+\frac{b^2}{4}-\frac{b^2}{4}+c}} = \int\frac{dx}{\sqrt{\left\{x^2+2x\left(\frac{b}{2}\right)+\frac{b^2}{4}\right\}-\frac{b^2}{4}+\frac{4c}{4}}}$
   $= \int\frac{dx}{\sqrt{\left\{x^2+2x\left(\frac{b}{2}\right)+\frac{b^2}{4}\right\}-\frac{b^2-4c}{4}}} = \int\frac{dx}{\sqrt{\left(x+\frac{b}{2}\right)^2-\frac{b^2-4c}{4}}}$, bila: $x+\frac{b}{2} = t$ dan $\frac{b^2-4c}{4} = \beta^2$
   Maka: $dx = dt$, sehingga: $\int\frac{dx}{\sqrt{\left(x+\frac{b}{2}\right)^2-\frac{b^2-4c}{4}}} = \int\frac{dt}{\sqrt{t^2-\beta^2}} =$
   $\frac{1}{\beta}\int\frac{\beta}{\sqrt{t^2-\beta^2}}dt$ dimana:
   $\frac{\beta}{\sqrt{t^2-\beta^2}} = \tan\theta$ dan $t = \beta\csc\theta = \frac{\beta}{\sin\theta} = \beta(\sin\theta)^{-1} \rightarrow dt = \beta\frac{d}{d\theta}(\sin\theta)^{-1}d\theta$

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$= \beta(-1)(\sin\theta)^{-2}\cos\theta\, d\theta = -\beta \frac{\cos\theta}{\sin^2\theta} d\theta$atau:$dt = -\beta \frac{\cos\theta}{\sin^2\theta} d\theta$

Sehingga:$\frac{1}{\beta}\int \frac{\beta}{\sqrt{t^2-\beta^2}} dt = \frac{1}{\beta}\int \tan\theta \left(-\beta \frac{\cos\theta}{\sin^2\theta}\right) d\theta = -\int \frac{d\theta}{\sin\theta} = -\ln\left(\frac{A\sin\theta}{1+\cos\theta}\right)$

$= \ln A\left(\frac{t+\sqrt{t^2-\beta^2}}{\beta}\right) = \ln A\left(\frac{x+\frac{b}{2}+\sqrt{\left(x+\frac{b}{2}\right)^2-\frac{b^2-4c}{4}}}{\frac{\sqrt{b^2-4c}}{2}}\right) = \ln A\left(\frac{2x+b+\sqrt{x^2+bx+\frac{b^2}{4}-\frac{b^2}{4}+c}}{\sqrt{b^2-4c}}\right) = \ln A\left(\frac{2x+b+\sqrt{x^2+bx+c}}{\sqrt{b^2-4c}}\right) = \ln \frac{A}{\sqrt{b^2-4c}}\left(2x+b+\sqrt{x^2+bx+c}\right)$

$= \ln B\left(2x+b+\sqrt{x^2+bx+c}\right) = \ln\left[B\left\{(2x+b)+\sqrt{x^2+bx+c}\right\}\right]$

Jadi:$\int \frac{dx}{\sqrt{x^2+bx+c}} = \ln\left[B\left\{(2x+b)+\sqrt{x^2+bx+c}\right\}\right]$

6. Tentukanlah: (a). $\int dx\sqrt{a+b\sin^2 x}$ dan (b). $\int dx\sqrt{a+b\cos^2 x}$
   Jawab:
   (a) $\int dx\sqrt{a+b\sin^2 x} = \int dx\sqrt{b\left(\frac{a}{b}+\sin^2 x\right)} = \sqrt{b}\int dx\sqrt{\frac{a}{b}+\sin^2 x}$

   Ambil: $\frac{a}{b}+\sin^2 x = u^2$, $2\sin x\cos x\, dx = 2u du = 2\sin x\, d(\sin x)$, sehingga:

   $\sin x = u$ dan$\cos x\, dx = du$, maka: $\sqrt{b}\int dx\sqrt{\frac{a}{b}+\sin^2 x} = \sqrt{b}\int \frac{du}{\cos x}\sqrt{u^2}$

   $= \sqrt{b}\int \frac{u}{\sqrt{1-u^2}} du = \sqrt{b}\int \tan x\cos x\, dx = \sqrt{b}\int \frac{\sin x}{\cos x}\cos x\, dx = \sqrt{b}\int \sin x\, dx$

   $= -\sqrt{b}(-\sin x)dx = -\sqrt{b}(C+\cos x) = -\sqrt{b}C - \sqrt{b}\cos x = D - \sqrt{b}\cos x$

   Jadi:$\int dx\sqrt{a+b\sin^2 x} = D - \sqrt{b}\cos x$

   (b) $\int dx\sqrt{a+b\cos^2 x} = \int dx\sqrt{b\left\{\frac{a}{b}+\cos^2 x\right\}} = \sqrt{b}\int dx\sqrt{\frac{a}{b}+\cos^2 x}$

   Ambil:$\frac{a}{b}+\cos^2 x = u^2$, $2\cos x(-\sin x)dx = 2\cos x\, d(\cos x) = 2u du$

   Sehingga: $\cos x = u$, $dx(-\sin x) = du$, $\sin x = \sqrt{1-u^2}$, $dx = \frac{du}{-\sin x} = \frac{du}{-\sqrt{1-u^2}}$

   Maka:$\sqrt{b}\int dx\sqrt{\frac{a}{b}+\cos^2 x} = \sqrt{b}\int \frac{du}{-\sqrt{1-u^2}}\sqrt{u^2} = -\sqrt{b}\int \frac{u}{\sqrt{1-u^2}} du$

   $= -\sqrt{b}\int \cot x(-\sin x)dx = \sqrt{b}\int dx \frac{\cos x}{\sin x}\sin x = \sqrt{b}\int \cos x\, dx$

   $= \sqrt{b}\int d(C+\sin x) = \sqrt{b}(C+\sin x) = C\sqrt{b}+\sqrt{b}\sin x = D+\sqrt{b}\sin x$

   Jadi:$\int dx\sqrt{a+b\cos^2 x} = D+\sqrt{b}\sin x$

7. Buktikanlah: (a). $\int dx\sqrt{a+b\sin x} = D + 2\sqrt{b}\left(\sin\frac{x}{2}-\cos\frac{x}{2}\right)$ dan (b). $\int dx\sqrt{a+b\cos x} = D+2\sqrt{2b}\sin\left(\frac{x}{2}\right)$
   Bukti:
   (a) $\int dx\sqrt{a+b\sin x} = \int\sqrt{b\left\{\frac{a}{b}+\cos\left(x-\frac{\pi}{2}\right)\right\}}\, d\left(x-\frac{\pi}{2}\right)$, ambil:

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$\left(\frac{a-b}{2b}\right) + \cos^2\left(\frac{x-\frac{\pi}{2}}{2}\right) = u^2$,

$2\cos\left(\frac{x-\frac{\pi}{2}}{2}\right)\left(-\sin\left(\frac{x-\frac{\pi}{2}}{2}\right)\right) d\left(\frac{x-\frac{\pi}{2}}{2}\right) = 2\cos\left(\frac{x-\frac{\pi}{2}}{2}\right) d\left(\cos\left(\frac{x-\frac{\pi}{2}}{2}\right)\right) = 2udu.$

Sehingga: $\cos\left(\frac{x-\frac{\pi}{2}}{2}\right) = u$, $\sin\left(\frac{x-\frac{\pi}{2}}{2}\right) = \sqrt{1-u^2}$, $-\sin\left(\frac{x-\frac{\pi}{2}}{2}\right) d\left(\frac{x-\frac{\pi}{2}}{2}\right) = du$

Atau: $-\sqrt{1-u^2}\, d\left(\frac{x-\frac{\pi}{2}}{2}\right) = du,\ d\left(\frac{x-\frac{\pi}{2}}{2}\right) = -\frac{du}{\sqrt{1-u^2}} = D + 2\sqrt{b}\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right)$

Jadi: $\int dx\sqrt{a + b\sin x} = D + 2\sqrt{b}\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right)$

(b) $\int dx\sqrt{a + b\cos x} = 2\sqrt{b}\int\sqrt{\left(\frac{a}{b} - 1\right) + 2\cos^2\left(\frac{x}{2}\right)}\, d\left(\frac{x}{2}\right)$

$= 2\sqrt{2b}\int\sqrt{\frac{a-b}{2b} + \cos^2\left(\frac{x}{2}\right)}\, d\left(\frac{x}{2}\right)$, ambil: $\frac{a-b}{2b} + \cos^2\left(\frac{x}{2}\right) = u^2$, maka:

$2\cos\frac{x}{2}\left(-\sin\frac{x}{2}\right) d\left(\frac{x}{2}\right) = 2\cos\frac{x}{2}\, d\left(\cos\frac{x}{2}\right) = 2udu$, sehingga: $\cos\frac{x}{2} = u$, $\sin\frac{x}{2} = \sqrt{1-u^2}$, $-\sin\frac{x}{2}\, d\left(\frac{x}{2}\right) = du$, $-\sqrt{1-u^2}\, d\left(\frac{x}{2}\right) = du$, $d\left(\frac{x}{2}\right) = -\frac{du}{\sqrt{1-u^2}}$

$2\sqrt{2b}\int\sqrt{\frac{a-b}{2b} + \cos^2\left(\frac{x}{2}\right)}\, d\left(\frac{x}{2}\right) = 2\sqrt{2b}\int\sqrt{u^2}\left(-\frac{du}{\sqrt{1-u^2}}\right) = -2\sqrt{2b}\int\frac{u}{\sqrt{1-u^2}} du$

$= 2\sqrt{2b}\int\cot\left(\frac{x}{2}\right)\sin\frac{x}{2}\, d\left(\frac{x}{2}\right) = 2\sqrt{2b}\int\frac{\cos\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)}\sin\frac{x}{2}\, d\left(\frac{x}{2}\right) = 2\sqrt{2b}\int\cos\left(\frac{x}{2}\right) d\left(\frac{x}{2}\right)$

$= 2\sqrt{2b}\left(C + \sin\left(\frac{x}{2}\right)\right) = 2\sqrt{2b}C + 2\sqrt{2b}\sin\left(\frac{x}{2}\right) = D + 2\sqrt{2b}\sin\left(\frac{x}{2}\right)$

Jadi: $\int dx\sqrt{a + b\cos x} = D + 2\sqrt{2b}\sin\left(\frac{x}{2}\right)$

8. Buktikanlah: $\int dx\sqrt{\frac{a}{x+c} + \frac{b}{x+d}} = D - 2\sqrt{(a+b)(x+c)}$

Bukti:

$\int dx\sqrt{\frac{a}{x+c} + \frac{b}{x+d}} = \int dx\sqrt{\frac{a(x+d)+b(x+c)}{(x+c)(x+d)}} = \int dx\sqrt{\frac{ax+ad+bx+bc}{x^2+(c+d)x+cd}} = \int dx\sqrt{\frac{(a+b)x+(ad+bc)}{x^2+(c+d)x+cd}}$

$= \int dx\sqrt{\frac{(a+b)\left(x+\frac{ad+bc}{a+b}\right)}{\left\{x^2+2\frac{c+d}{2}x+\left(\frac{c+d}{2}\right)^2\right\}-\left\{\left(\frac{c+d}{2}\right)^2-cd\right\}}} = \int dx\frac{\sqrt{a+b}\sqrt{x+\frac{ad+bc}{a+b}}}{\sqrt{\left(x+\frac{c+d}{2}\right)^2-\left(\frac{c-d}{2}\right)^2}}$

$= \sqrt{a+b}\int dx\frac{\sqrt{\left(x+\frac{c+d}{2}\right)-\left(\frac{ac+3bc+3ad+bd}{2(a+b)}\right)}}{\sqrt{\left(x+\frac{c+d}{2}\right)^2-\left(\frac{c-d}{2}\right)^2}} = \sqrt{a+b}\int dx\frac{\sqrt{u-B}}{\sqrt{u^2-A^2}} = \sqrt{a+b}\int du\frac{\sqrt{A}\sqrt{C-\frac{u}{A}}}{\frac{\sqrt{A^2-u^2}}{A}A}$

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$= \sqrt{\frac{a+b}{A}} \int du \frac{\sqrt{C-\frac{u}{A}}}{\frac{\sqrt{A^2-u^2}}{A}}$ dimana: $x + \frac{c+d}{2} = u$, $\frac{c-d}{2} = A$ dan $\frac{ac+3bc+3ad+bd}{2(a+b)} = B$, $\frac{B}{A} =$
C, sehingga:

$dx = du$, $A\cos\theta = u$, $-A\sin\theta\, d\theta = du$, maka: $\sqrt{\frac{a+b}{A}} \int du \frac{\sqrt{C-\frac{u}{A}}}{\frac{\sqrt{A^2-u^2}}{A}}$

$= -A\sqrt{\frac{a+b}{A}} \int \sin\theta\, d\theta \frac{\sqrt{C-\cos\theta}}{\sin\theta} = \sqrt{A(a+b)} \int d\theta \sqrt{C-\cos\theta} = D -$
$2\sqrt{(a+b)(x+c)}$

Jadi: $\int dx \sqrt{\frac{a}{x+c} + \frac{b}{x+d}} = D - 2\sqrt{(a+b)(x+c)}$

Menghitung panjang lintasan pada koordinat kartesian

Dengan menggunakan metoda: $s = \int_A^B \sqrt{\delta_{ij}\dot{x}^i\dot{x}^j}\, dt$ atau: $s = \int_A^B \sqrt{\delta_{ij}v^i v^j}\, dt$, maka diperolehpenyelesaian dari contoh soal sbb:

1. Partikel bergerakdengan posisi setiap saatnya adalah: $x = \sin 2t$, $y = 2\sin 2t$ dan $z = 3\cos 2t$ meter, tentukanlah lintasannya dari $t = \frac{\pi}{5}$ detik ke $t = \frac{\pi}{3}$ detik.
   Jawab:
   $s = \int_{t=\frac{\pi}{5}}^{\frac{\pi}{3}} \sqrt{\delta_{ij}\dot{x}^i\dot{x}^j}\, dt = \int_{t=\frac{\pi}{5}}^{\frac{\pi}{3}} \sqrt{\delta_{jj}\dot{x}^j\dot{x}^j}\, dt = \int_{t=\frac{\pi}{5}}^{\frac{\pi}{3}} \sqrt{\dot{x}^j\dot{x}^j}\, dt$
   $= \int_{t=\frac{\pi}{5}}^{\frac{\pi}{3}} \sqrt{\dot{x}^1\dot{x}^1 + \dot{x}^2\dot{x}^2 + \dot{x}^3\dot{x}^3}\, dt = \int_{t=\frac{\pi}{5}}^{\frac{\pi}{3}} \sqrt{(\dot{x}^1)^2 + (\dot{x}^2)^2 + (\dot{x}^3)^2}\, dt$
   $= \int_{t=\frac{\pi}{5}}^{\frac{\pi}{3}} \sqrt{(\dot{x})^2 + (\dot{y})^2 + (\dot{z})^2}\, dt = \int_{t=\frac{\pi}{5}}^{\frac{\pi}{3}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$
   $= \int_{t=\frac{\pi}{5}}^{\frac{\pi}{3}} \sqrt{\left(\frac{d}{dt}(\sin 2t)\right)^2 + \left(\frac{d}{dt}(2\sin 2t)\right)^2 + \left(\frac{d}{dt}(3\cos 2t)\right)^2} \frac{1}{2}\left(d(2t)\right)$
   $= \frac{1}{2}\int_{t=\frac{\pi}{5}}^{\frac{\pi}{3}} \sqrt{(2\cos 2t)^2 + (4\cos 2t)^2 + (6\sin 2t)^2}\, d(2t) = 2\{\sin(60^0 + \pi) -$
   $\sin(72^0 + \pi)\}$
   $= 2\{-\sin 60^0 + \sin 72^0\} = 2\{-0{,}866 + 0{,}95\} = 0{,}169668$ meter. Jadi: $s =$
   0,169668
2. Partikel bergerak dengan posisi setiap saatnya adalah: $x = \sin t$, $y = \sin 2t$ dan $z = \cos 3t$ meter, tentukanlah lintasannya dari $t = \frac{\pi}{5}$ detik ke $t = \frac{\pi}{3}$ detik.
   Jawab:
   $s = \int_{t=\frac{\pi}{5}}^{\frac{\pi}{3}} \sqrt{\delta_{ij}\dot{x}^i\dot{x}^j}\, dt = \int_{t=\frac{\pi}{5}}^{\frac{\pi}{3}} \sqrt{\delta_{jj}\dot{x}^j\dot{x}^j}\, dt = \int_{t=\frac{\pi}{5}}^{\frac{\pi}{3}} \sqrt{\dot{x}^j\dot{x}^j}\, dt =$
   $\int_{t=\frac{\pi}{5}}^{\frac{\pi}{3}} \sqrt{\dot{x}^1\dot{x}^1 + \dot{x}^2\dot{x}^2 + \dot{x}^3\dot{x}^3}\, dt$

$= \int_{t=\frac{\pi}{5}}^{\frac{\pi}{3}} \sqrt{(\dot{x}^1)^2 + (\dot{x}^2)^2 + (\dot{x}^3)^2} dt = \int_{t=\frac{\pi}{5}}^{\frac{\pi}{3}} \sqrt{(\dot{x})^2 + (\dot{y})^2 + (\dot{z})^2} dt =$
$\int_{t=\frac{\pi}{5}}^{\frac{\pi}{3}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt =$
$\int_{t=\frac{\pi}{5}}^{\frac{\pi}{3}} \sqrt{\left(\frac{d}{dt}(\sin t)\right)^2 + \left(\frac{d}{dt}(\sin 2t)\right)^2 + \left(\frac{d}{dt}(\cos 3t)\right)^2} dt$
$= \int_{t=\frac{\pi}{5}}^{\frac{\pi}{3}} \sqrt{(\cos t)^2 + (2\cos 2t)^2 + \left(\frac{d}{dt}(\cos t)\right)^2} dt =$
$\int_{t=\frac{\pi}{5}}^{\frac{\pi}{3}} \sqrt{\cos^2 t + 4\cos^2 2t + \sin^2 t} dt$
$= \int_{t=\frac{\pi}{5}}^{\frac{\pi}{3}} \sqrt{(\cos^2 t + \sin^2 t) + 4\cos^2 2t} dt = \frac{1}{2}\int_{t=\frac{\pi}{5}}^{\frac{\pi}{3}} \sqrt{1 + 4\cos^2(2t)} d(2t)$
$= \int_{t=\frac{\pi}{5}}^{\frac{\pi}{3}} \sqrt{\frac{1}{4} + \cos^2(2t)} d(2t) = \int_{t=\frac{\pi}{5}}^{\frac{\pi}{3}} \sqrt{\frac{1}{4} + \cos^2(2t)} d(2t)$, ambil: $\frac{1}{4} + \cos^2(2t) = u^2$, sehingga:S $= -0{,}0843$ meter.

3. Peluru ditembakkan diatas bidang datar dengan laju kecepatan awal $v_0 = 200 \frac{meter}{detik}$ dan sudut elevasi $\theta = 37^0$, bila laju percepatan gravitasi bumi g = 10 $\frac{meter}{detik^2}$, tentukanlah panjang lintasan peluru diudara sebelum jatuh ketanah$s$.
   Jawab:
   Uraian laju kecepatan peluru:
   $v_{0y} = v_0 \cos\theta = 200\cos 37^0 = 200 \times \frac{4}{5} = 160 \frac{meter}{detik} = v_y$
   $v_{0z} = v_0 \sin\theta = 200 \sin 37^0 = 200 \times \frac{3}{5} = 120 \frac{meter}{detik}$, maka: $v_z = v_{0z} - gt$
   $= (120 - 10t) \frac{meter}{detik}$, pada titik puncak A:$v_{zA} = 0 = 120 - 10t_{naik} = 0 \rightarrow t_{naik}$
   $= 12$ detik $= t_{turun}$, $t_{total} = t_{naik} + t_{turun} = 12 + 12 = 24$ detik $= \Delta t = t_2 - t_1$, sehingga:
   $s = \int_{t=0}^{24} \sqrt{\delta_{ij}\dot{x}^i\dot{x}^j} dt = \int_{t=0}^{24} \sqrt{\delta_{jj}\dot{x}^j\dot{x}^j} dt = \int_{t=0}^{24} \sqrt{\dot{x}^j\dot{x}^j} dt = \int_{t=0}^{24} \sqrt{\dot{x}^2\dot{x}^2 + \dot{x}^3\dot{x}^3} dt$
   $= \int_{t=0}^{24} \sqrt{(\dot{x}^2)^2 + (\dot{x}^3)^2} dt = \int_{t=0}^{24} \sqrt{(\dot{y})^2 + (\dot{z})^2} dt = \int_{t=0}^{24} \sqrt{v_y^2 + v_z^2} dt$
   $= \int_{t=0}^{24} \sqrt{(160)^2 + (120 - 10t)^2} dt =$
$\int_{t=0}^{24} \sqrt{25600 + 14400 - 2400t + 100t^2} dt$
   $= \int_{t=0}^{24} \sqrt{40000 - 2400t + 100t^2} dt = 10\int_{t=0}^{24} \sqrt{400 - 24t + t^2} dt$
   $= 10\int_{t=0}^{24} \sqrt{(400 - 144) + (144 - 2 \times 12t + t^2)} dt =$
$10\int_{t=0}^{24} \sqrt{256 + (12 - t)^2} dt$
   $= 10\int_{t=0}^{24} \sqrt{(t - 12)^2 + 256} dt = 10(I|_{t=0}^{24})$, dimana: $I = \int \sqrt{(t - 12)^2 + 256} dt$ =
   $= \frac{(t-12)\sqrt{(t-12)^2+256}}{2} + 78\ln\left|\frac{(t-12)+\sqrt{(t-12)^2+256}}{16}\right|$, maka panjang lintasannya S = $10(I|_{t=0}^{24})$
   $= 10\{16 + 78\ln 2 + 120 + 78\ln 2\} = 10\{136 + 78\ln 4\} = 1095{,}173$ meter.
4. Peluru ditembakkan diatas bidang miring dengan sudut kemiringan $\alpha = 37^0$ yang menuju kebawah dengan laju kecepatan awal $v_0 = 200 \frac{meter}{detik}$ dan sudut elevasi θ

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= 37^0, bila laju percepatan gravitasi bumi g = 10 $\frac{meter}{detik^2}$, tentukanlah panjang lintasan peluru diudara sebelum jatuh ketanah s.
Jawab:
Uraian laju kecepatan peluru:
$v_{0y} = v_0 \cos\theta = 200\cos 37^0 = 200 \times \frac{4}{5} = 160\,\frac{meter}{detik} = v_y$
$v_{0z} = v_0 \sin\theta = 200\sin 37^0 = 200 \times \frac{3}{5} = 120\,\frac{meter}{detik}$, maka: $v_z = (120 - 10t)\,\frac{meter}{detik}$, pada titik puncak A: $v_{zA} = 0 = 120 - 10t_{naik} = 0 \rightarrow t_{naik}$ = 12 detik
$A'A = v_{0z}t_{naik} - \frac{1}{2}gt_{naik}^2 = 120 \times 12 - \frac{1}{2} \times 10 \times 144 = 1440 - 720 = 720$ meter, $CD = OC\tan\alpha = v_y t_{total}\tan 37^0 = 120(12 + t_{turun})\frac{3}{4} = 90(12 + t_{turun}) = 1080 + 90t_{turun}$, $BD = \frac{1}{2}gt_{turun}^2 = 5t_{turun}^2$, $CD = BD - BC$, $BD = CD + BC$. Atau $5t_{turun}^2$
$= 1080 + 90t_{turun} + A'A = 1080 + 90t_{turun} + 720 = 1800 + 90t_{turun}$, atau:
$t_{turun}^2 - 18t_{turun} - 360 = 0$, $(t_{turun}^2 - 2 \times 9t_{turun} + 81) - (81 + 360) = 0$
$(t_{turun} - 9)^2 - 400 = 0$, $(t_{turun} - 9)^2 = 400$, $t_{turun} - 9 = \pm 20$, ambil: t_{turun} = 29 detik dan $t_{total} = t_{naik} + t_{turun}$ = 12 + 29 = 41 detik.
Sehingga: $s = \int_{t=0}^{41}\sqrt{\delta_{jj}\dot{x}^j\dot{x}^j}dt = \int_{t=0}^{41}\sqrt{\dot{x}^j\dot{x}^j}dt = \int_{t=0}^{41}\sqrt{\dot{x}^2\dot{x}^2 + \dot{x}^3\dot{x}^3}dt$
$= \int_{t=0}^{41}\sqrt{(\dot{x}^2)^2 + (\dot{x}^3)^2}dt = \int_{t=0}^{41}\sqrt{(\dot{y})^2 + (\dot{z})^2}dt = \int_{t=0}^{41}\sqrt{v_y^2 + v_z^2}dt$
$= \int_{t=0}^{41}\sqrt{25600 + (120 - 10t)^2}dt =$
$\frac{(t-12)\sqrt{(t-12)^2+256}}{2} + 78\ln\left|\frac{(t-12)+\sqrt{(t-12)^2+256}}{16}\right|$
Sehingga panjang lintasan adalah:S = $10(I|_{t=0}^{41}) = 10\{600{,}24 + 78\ln 7{,}76\}$ = 2198,61meter

PENUTUP

Simpulan

Dari tulisan diatas maka terdapat beberapa simpulan sebagai berikut:

1) $\int\frac{dx}{\cos^5 x} = \frac{\sin x}{4\cos^4 x} + \frac{3\sin x}{8\cos^2 x} + \frac{3}{4}\ln\sqrt{C\left|\frac{1+\sin x}{\cos x}\right|}$
2) $\int\frac{dx}{\sin^5 x} = -\frac{\cos x}{4\sin^4 x} - \frac{3}{8}\frac{\cos x}{\sin^2 x} + \frac{3}{4}\ln\sqrt{C\left|\frac{\sin x}{1+\cos x}\right|}$
3) $\int\frac{a}{x+b}dx = D + \ln|x+b|^a$
4) $\int\frac{a}{x^2-b^2}dx = \ln\left|\frac{x-b}{x+b}\right|^{\frac{a^2}{2b}} + C$ dan $\int\frac{a}{x^2+b^2}dx = C - \frac{a}{b}\cot^{-1}\left(\frac{x}{b}\right)$
5) $\int\frac{a}{x^2+bx+c}dx = \ln\left|\frac{2x+b-\sqrt{b^2-4c}}{2x+b+\sqrt{b^2-4c}}\right|^{\sqrt{b^2-4c}} + C$
6) $\int\sqrt{a^2-x^2}\,dx = \frac{x\sqrt{a^2-x^2}}{2} - \frac{a^2}{2}\cos^{-1}\left(\frac{x}{a}\right) + C$
7) $\int\sqrt{x^2+4x+3}\,dx = \ln\left|C\frac{1}{\sqrt{(x+2)+\sqrt{x^2+4x+3}}}\right|$
8) $\int\sqrt{x^2-a^2}\,dx = \frac{x\sqrt{x^2-a^2}}{2} - a^2\ln\frac{\sqrt{x+\sqrt{x^2-a^2}}}{B}$

9) $\int dx\sqrt{a + b\cos x} = D + 2\sqrt{2b}\sin\left(\frac{x}{2}\right)$

10) $\int dx\sqrt{\frac{a}{x+c} + \frac{b}{x+d}} = D - 2\sqrt{(a+b)(x+c)}$

Saran

Berdasarkan dari uraian dan kesimpulan diatas maka penulis menyarankan kepada rekan-rekan pengajar di prodi Pendidikan Fisika khususnya mata kuliah Kalkulus terutama dalam hal memberikan contoh soal tentang soal-soal integral supaya tidak semata-mata langsung diberikan berdasarkan tabel integral saja saja, jadi cobalah diberikan uraian teknik-teknik penyelesaian pengintegralannya agar supaya para mahasiswa dapat melihat dan belajar sendiri pengerjaan soalnya tidak berdasarkan tabel semata. Ingat matematika itu bukan doktrin.

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